Q1(a), January 2021

Find all possible Jordan Canonical Forms of a 5 x 5 matrix A whose cube is zero.

Observations:

- The Jordan Canonical Form is built from the elementary divisors of A.
- The elementary divisors of A are obtained from the invariant factors of A.
- The invariant factors of A must each divide the minimal polynomial of A.
- The minimal polynomial of A is the largest invariant factor.

Smith Normal Form

Big Brain Observation: it suffices to find the minimal polynomial of A.

$A^3 = 0 \Longrightarrow \mu_A(x) \mid x^3 \Longrightarrow$ The minimal polynomial is x or x^2 or x^3.

1.) The minimal polynomial is x. So, the invariant factors are all x. The elementary divisors are the powers of irreducible polynomials dividing the invariant factors (which are all x), so the elementary divisors must be x. This is a 5 x 5 matrix, so the elementary divisors are x, x, x, x, and x. The Jordan Canonical Form is the direct sum of the Jordan blocks of the elementary divisors (which are all x), and the Jordan block corresponding to x is (0). So, JCF(A) = 0.

2.) The minimal polynomial is x^2. The invariant factors are non-contstant polynomials that must all divide the minimal polynomial, so the only possible invariant factors are x and x^2.

a.) If the invariant factors are x^2, x^2, and x, then the elementary divisors are x^2, x^2, and x. The Jordan Canonical Form is $J_{x^2} \oplus J_{x^2} \oplus J_x$, i.e.,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & 0 & \\ & & 0 & 1 & \\ & 0 & 0 & 0 & \\ & & & & 0 \end{pmatrix}.$$

b.) If the invariant factors are x^2, x, x, and x, so the elementary divisors are x^2, x, x, and x. The Jordan Canonical Form is $J_{x^2} \oplus_{i = 1}^3 J_x$, i.e.,

$$\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & 0 & \\ & & 0 & & \\ & 0 & & 0 & \\ & & & & 0 \end{pmatrix}.$$

3.) The minimal polynomial is x^3.

a.) The invariant factors are x^3 and x^2.

b.) The invariant factors are x^3, x, and x.

In either case, the elementary divisors and the invariant factors coincide.
In general, if $p(x)$ is an invariant factor with factorization into powers of distinct irreducibles

$$p(x) = q_1(x)^{e_1} \dots q_k(x)^{e_k},$$

then the elementary divisors are $q_1(x)^{e_1}, \dots, q_k(x)^{e_k}$.

Ex.: If $p(x) = x^3$ is an invariant factor, then $q(x) = x^3$ is an elementary divisor.

Ex.: If the real polynomial $p(x) = (x^2 - 4)(x^2 + 1)$ is an invariant factor, then $q_1(x) = x - 2$, $q_2(x) = x + 2$, and $q_3(x) = x^2 + 1$ are the elementary divisors.

Q3(b.), January 2021

Prove that if A^2 = 3A, then A is similar to a diagonal matrix whose diagonal entries are 0 or 3.

Observation:

If A^2 = 3A, then A^2 - 3A = 0 so that A(A - 3I) = 0. The minimal must divide x(x - 3).

The upshot is that the invariant factors of A must be x, x - 3, or x(x - 3). Therefore, the elementary divisors of A can only be x and x - 3, hence A is similar to its Jordan Canonical Form, which is a diagonal matrix with diagonal entries of 0 or 3. QED.

Q1, January 2020

Describe all possible Jordan Canonical Forms of a 4 x 4 matrix whose minimal polynomial has only two (possibly repeated) irreducible factors. Give an example in each case.

Let x - a and x - b be the irreducible factors of the minimal polynomial of A.

1.) The minimal polynomial is (x - a)(x - b).

a.) If x - a is an invariant factor, it must appear exactly twice, i.e., (x - a) | (x - a) | (x - a)(x - b).

The elementary divisors are x - a, x - a, x - a, and x - b.

$$JCF(A) = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & b \end{pmatrix} \longleftarrow \text{example}$$

b.) If x - b is an invariant factor, then

$$JCF(A) = \begin{pmatrix} a & & & 0 \\ & b & & \\ & & b & \\ 0 & & & b \end{pmatrix}. \longleftarrow \text{example}$$

c.) If (x - a)(x - b) is the only invariant factor, then

$$JCF(A) = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & b & \\ 0 & & & b \end{pmatrix}.$$

2.) The minimal polynomial is (x - a)^2 (x - b).

a.) If x - a is an invariant factor, then the invariant factors are x - a and (x - a)^2 (x - b). The elementary divisors are x - a, (x - a)^2, and x - b. The Jordan Canonical Form is as follows.

$$JCF = \begin{pmatrix} a & & & \\ & a & 1 & \\ & 0 & a & \\ & & & b \end{pmatrix}.$$

b.) If x - b is an invariant factor, then the elementary divisors are x - b, (x - a)^2, and x - b.

$$JCF = \begin{pmatrix} b & & & \\ & a & 1 & \\ & 0 & a & \\ & & & b \end{pmatrix}$$

3.) If the minimal polynomial is (x - a)(x - b)^2, then we refer to case (2.).

4.) If the minimal polynomial is (x - a)^2 (x - b)^2, then the characteristic polynomial is (x - a)^2 (x - b)^2, so the elementary divisors are (x - a)^2 and (x - b)^2.

$$JCF = \begin{pmatrix} a & 1 & & \\ 0 & a & & \\ & & b & 1 \\ & & 0 & b \end{pmatrix}$$

Describe the precise circumstances under which the Jordan Canonical Form of an n x n matrix is equal to its Rational Canonical Form.

Recall that the Rational Canonical Form is built from the companion matrices of the invariant factors. In particular, the RCF (typically) has 1s on the subdiagonal. On the other hand, the Jordan Canonical Form is built from the Jordan blocks of the elementary divisors. In particular, the JCF (typically) has 1s on the superdiagonal. Consequently, they are equal if and only if both are diagonal if and only if the elementary divisors are all linear and the invariant factors are all linear if and only if the invariant factors are all linear if and only if the minimal polynomial is a linear polynomial if and only if A = aI for some complex number a.